

Deterministic Weighted Scale-free Small-world Networks

Yichao Zhang^a Zhongzhi Zhang^{b,c} Shuigeng Zhou^{b,c}
Jihong Guan^{a,*}

^a*Department of Computer Science and Technology, Tongji University,
4800 Cao'an Road, Shanghai 201804, China*

^b*Department of Computer Science and Engineering, Fudan University,
Shanghai 200433, China*

^c*Shanghai Key Lab of Intelligent Information Processing, Fudan University,
Shanghai 200433, China*

Abstract

We propose a deterministic weighted scale-free small-world model for considering pseudofractal web with the coevolution of topology and weight. Considering the fluctuations in traffic flow constitute a main reason for congestion of packet delivery and poor performance of communication networks, we suggest a recursive algorithm to generate the network, which restricts the traffic fluctuations on it effectively during the evolutionary process. we provide a relatively complete view of topological structure and weight dynamics characteristics of the networks: weight and strength distribution; degree correlations; average clustering coefficient and degree-cluster correlations; as well as the diameter.

Key words: Complex networks, Scale-free networks, Weighted networks, Disordered systems, Traffic fluctuations

1 Introduction

To understand the general principles in architectures of networks, many deterministic models are introduced into complex networks [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,

* Corresponding author.

Email addresses: zhangzz@fudan.edu.cn (Zhongzhi Zhang),
sgzhou@fudan.edu.cn (Shuigeng Zhou), jhguan@mail.tongji.edu.cn (Jihong Guan).

These models are useful tools for investigating analytically not only topological features of networks in detail [1,2,3,4,5,6,7,8,9,10,11,12], but also dynamical problems on the networks [13,14,15,16]. Before presenting our own findings, it is worth reviewing some of this preceding work to understand its achievements and shortcomings. Deterministic scale-free networks were firstly proposed by Barabási *et al.* in Ref. [1] and intensively studied in Ref. [2] to generate a scale-free topology. However, to some extent, the small exponent γ of the degree distribution for the model didn't satisfy the real statistic results well. Instead, Dorogovtsev *et al.* introduced another elegant model, called pseudofractal scale-free web (PSW) [3] which is extended by Comellas *et al.* consequently [4]. Based on a similar idea of PSW, Jung *et al.* presented a class of recursive trees [5], which have the small-world behavior built in. Additionally, in order to discuss modularity, Ravasz *et al.* proposed a hierarchical network model [6,7], the exact scaling properties and extensive study of which were reported in Refs. [8] and [9], respectively. Recently, motivated by the problem of Apollonian space-filling packing, Apollonian networks [10] with a typical loop structure were introduced and intensively investigated [17,18,19,20,21]. These pioneering works are all invaluable tools for the topology of networks studies.

In the last few years, it is found that many real networks are inhomogeneous, consisting of distinct nodes and links. For instance, the scientist collaboration network, where scientists are identified with nodes, and an edge exists between two scientists if they have coauthored at least one paper [22], and the Internet at the AS level, where the link weights represent the bandwidth of a cable and node weight the load of a router [23], among other areas. Recently, weight dynamics ideas have been applied with success to topics as diverse, such as random walks [24], condensation [25], synchronization [26], traffic congestion [27], epidemic spreading [28,29], information filtering [30], to name but a few. The findings above might provide insight for understanding the correlations among weighted quantities and the underlying topological structure and dynamics behaviors of the weighted networks.

Most previous weighted random models [31,32,33,34,35] with topology and weight coevolution, however, possess very loose clustering structures when the size of the networks is large. At the same time, previous deterministic models, are mainly unweighted [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21], which ignore the heterogeneity of edges in real networks. What's more, the models [3,4] on PSW networks fail to provide the reason for adopting the recursive way to build up the networks. Consequently, in this paper, we introduce a model bringing weight evolution into the growth of pseudofractal scale-free web (PSW) [3] that aims to circumvent these incongruities properly. As we will show, in the case of the recursive construction, the traffic and its fluctuation decrease exponentially with time either on edges or at nodes. Hence, we believe the construction method may shed some light on networks design to improve the control and speed of the whole network [36]. At the same time,

our comprehensive and rigorous solutions may help people understand better the interplay between network topology and weight dynamics.

2 The model

The construction of the model is controlled by two parameters m and δ , evolving in a recursive way. We denote the network after t steps by $G(t)$, $t \geq 0$ (see Fig. 1). Then the network at step t is constructed as follows. For $t = 0$, $R(0)$ is a triangle consisting of three links with unit weight. For $t \geq 1$, $G(t)$ is obtained from $G(t - 1)$. We add mw (m is positive integer) new nodes for each of the links with weight w , and we connect each new node to both ends of this link by new links of unit weight; moreover, we increase the weight of these links by $m\delta w$ (δ is positive integer).

Before introducing our model further, we explain why adopting such a recursive way and why the generated network is increasingly efficient for transmitting information with network order. In this model, the recursive construction is motivated by the practical need to improve the transport capacity of real networks. As is known to us, both the physical networks and the numbers of users are growing continuously. The performance of the networks for larger system sizes and heavier loads are critical issues to be addressed in order to guarantee networks' functioning in the near future. For example, if the traffic fluctuates dramatically, a highway is more likely to be congested frequently when the peak value of traffic exceeds its capacity. There is thus a need to build up more branches to distribute the heavy traffic. However, "how many branches must we have?" and "Where shall we put them?" are open questions yet.

Recently, the authors of Ref. [37] claimed the fluctuations in traffic flow constitute the main factor affecting the performance of networks. They derived the dependence of fluctuations with the mean traffic on unweighted networks analytically. Consequently, their recipes were adopted extensively to the weighted networks by the authors of Ref [38]. As shown in Ref [38], for correlated networks (assortative or disassortative mixing [39,40]), the average traffic through a link L_{ij} during a time window can be represented as

$$\langle f_{ij} \rangle = \frac{2w_{ij}}{\sum_{i=1}^N s_i} RM, \quad (1)$$

and the standard deviation can be expressed as

$$\sigma_{ij}^2 = \langle f_{ij} \rangle \left(1 + \langle f_{ij} \rangle \frac{\Delta^2 + \Delta}{3R^2} \right), \quad (2)$$

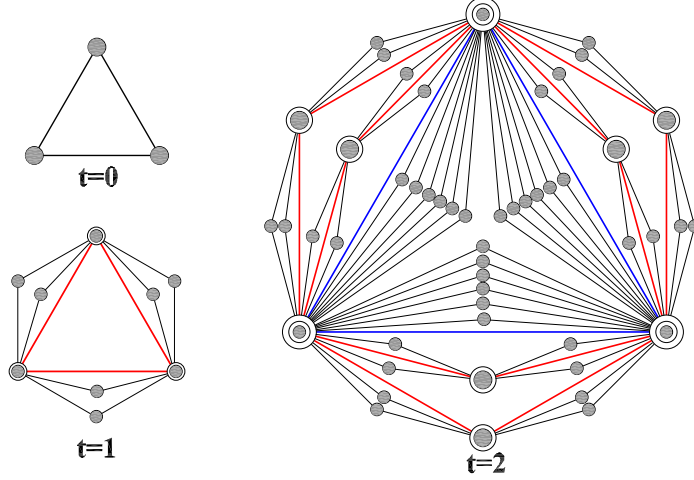


Fig. 1. Illustration of the deterministically growing network for the particular case of $m = 2$ and $\delta = 1$, showing the first three steps of growing process. The gray links in the figure denote the links with weight 1, the red links with weight 3, and the blue links with weight 9. The number of rings around a gray node denotes its age.

where w_{ij} is its link weight [38]. The length of the time window for observation is M . The average number of cars or walkers among various time windows in the network is denoted by R . Δ is defined as a random variable representing the number of walkers travelling through the link in the time window.

With respect to the traffic at nodes, the average traffic at a node i

$$\langle f_i \rangle = \frac{s_i}{\sum_{i=1}^N s_i} RM, \quad (3)$$

Then the standard deviation as a function of $\langle f_i \rangle$ can be obtained as:

$$\sigma_i^2 = \langle f_i \rangle \left(1 + \langle f_i \rangle \frac{\Delta^2 + \Delta}{3R^2} \right). \quad (4)$$

At each time step, the traffic will be distributed dispersedly to the newly built edges (nodes). The larger, the size of the considered network is, the lower the fluctuations on each edge (at each node) should be. Details of the analysis are provided in section 3.1 and 3.2.

Notice that there are in fact three limiting cases of the present model. In the special case $m = 1$ and $\delta = 0$, it is reduced to the pseudofractal scale-free web described in [3]. When $\delta = 0$, it is a particular case of the geometric growth networks discussed in [41]. When $m = 1$, it is the same as the deterministic weighted networks proposed in [42]. Thus, vary parameters m and δ , we can study many crossovers between these limiting cases.

Let us consider the total number of nodes N_t , the total number of links E_t and the total weight of all links W_t in $G(t)$. The number of nodes created at

step t is denoted by $n_v(t)$. Note that the addition of each new node leads to two new links, so the number of links generated at step t is $n_e(t) = 2n_v(t)$. By construction, for $t \geq 1$, we have

$$n_v(t) = mW_{t-1}, \quad (5)$$

$$E_t = E_{t-1} + 2n_v(t), \quad (6)$$

and

$$W_t = W_{t-1}(1 + m\delta) + 2mW_{t-1}. \quad (7)$$

On the right-hand side of Eq. (7), the first item is the sum of weight of the old links, and the second term describes the total weight of the new links generated in step t . Eq. (7) can be simplified to

$$W_t = (1 + m\delta + 2m)W_{t-1}. \quad (8)$$

Considering the initial condition $W_0 = 3$, we obtain

$$W_t = 3(1 + m\delta + 2m)^t. \quad (9)$$

Substituting Eq. (9) into Eq. (5) and using $W_0 = 3$, the number of nodes created at step t ($t \geq 1$) is obtained to be

$$n_v(t) = 3m(1 + m\delta + 2m)^{t-1}. \quad (10)$$

Hence, one can figure out the growth of the network is accelerated. Then the total number of nodes present at step t is

$$N_t = \sum_{t_i=0}^t n_v(t_i) = \frac{3}{2 + \delta} \left[(1 + m\delta + 2m)^t + \delta + 1 \right]. \quad (11)$$

Combining Eq. (10) with Eq. (6) and considering $E_0 = 3$, it follows that

$$E_t = \frac{3}{2 + \delta} \left[(1 + m\delta + 2m)^t + \frac{\delta}{2} \right]. \quad (12)$$

Thus for large t , the average degree $\bar{k}_t = \frac{2E_t}{N_t}$ is approximately equal to 4.

3 Structural properties

In what follows we will study how the tunable parameters m and δ control some relevant characteristics of the weighted network $G(t)$. Firstly, we give out the analytical solution of distribution of strength, degree and weight to test its scale-free nature; simultaneously, we show the analytical expression of

average traffic and its deviation of nodes and edges; subsequently, we move forward to the average clustering coefficient coupled with the diameter of this network for the sake of verifying its small-world property; finally, we study the degree correlations as well.

3.1 Weight distribution

Let $w_e(t)$ be the weight of link e at step t . In the view of all the links emerging simultaneously have the same weight, it can be recast recursively as follows

$$w_e(t) = (1 + m\delta)w_e(t - 1). \quad (13)$$

If link e enters the network at step τ , then $w_e(\tau) = 1$. Thus, we can easily have

$$w_e(t) = (1 + m\delta)^{t-\tau}. \quad (14)$$

Obviously, the weight spectrum of the network is discrete. It follows that the weight distribution is given by

$$P(w) = \begin{cases} \frac{n_e(0)}{E_t} = \frac{2 + \delta}{2(1 + m\delta + 2m)^t + \frac{\delta}{2}} & \text{for } \tau = 0, \\ \frac{n_e(\tau)}{E_t} = \frac{2m(2 + \delta)(1 + m\delta + 2m)^{\tau-1}}{2(1 + m\delta + 2m)^t + \frac{\delta}{2}} & \text{for } \tau \geq 1, \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

and that the cumulative weight distribution [43,3] is

$$P_{\text{cum}}(w) = \sum_{\mu \leq \tau} \frac{n_e(\mu)}{E_t} = \frac{2(1 + m\delta + 2m)^\tau + \frac{\delta}{2}}{2(1 + m\delta + 2m)^t + \frac{\delta}{2}}. \quad (16)$$

Substituting for τ in this expression using $\tau = t - \frac{\ln w}{\ln(1+m\delta)}$ gives

$$\begin{aligned} P_{\text{cum}}(w) &= \frac{2(1 + m\delta + 2m)^t w^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta)}} + \frac{\delta}{2}}{2(1 + m\delta + 2m)^t + \frac{\delta}{2}} \\ &\approx w^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta)}} \quad \text{for large } t. \end{aligned} \quad (17)$$

Apparently, the weight distribution follows a power law with the exponent $\gamma_w = 1 + \frac{\ln(1+m\delta+2m)}{\ln(1+m\delta)}$. For the particular case of $m = 1$, Eq. (17) recovers the result previously obtained in Ref. [42].

In this paper, the networks generated by the model are disassortative, which will be discussed analytically in section 3.6. For disassortative networks, the

fluctuation σ_{ij} on an edge L_{ij} relies on average traffic f_{ij} , while f_{ij} is governed by $\frac{2w_{ij}}{\sum_{i=1}^N s_i}$ with R and M fixed. Inserting Eq. (14) and Eq. (9) into Eq. (1), one can obtain the average traffic on an arbitrary edge L_{ij} can be written as

$$\langle f_{ij} \rangle = \frac{2w_{ij}(t)}{2W_t} RM = \frac{(1+m\delta)^{t-\tau_{ij}}}{3(1+m\delta+2m)^t} RM, \quad (18)$$

where τ_{ij} denotes the entry time of L_{ij} . Thus the standard deviation can be expressed as

$$\begin{aligned} \sigma_{ij}^2 = & \left(\frac{1+m\delta}{1+m\delta+2m} \right)^t \frac{(1+m\delta)^{-\tau_{ij}} RM}{3} \\ & + \left(\frac{1+m\delta}{1+m\delta+2m} \right)^{2t} \frac{(1+m\delta)^{-2\tau_{ij}} (\Delta^2 + \Delta) M^2}{27}, \end{aligned} \quad (19)$$

Eq. (18) and Eq. (19) show that both f_{ij} and σ_{ij} decrease as an exponential function of time t with R and W fixed as a result of $\frac{1+m\delta}{1+m\delta+2m} < 1$. Notice that, we consider the parameter R as a const with the time evolution in our model in that the relation between $R(t)$ and t depends on various specific systems, which is not the focus of the present paper. In fact, one can easily observe that the recursive algorithm can also restrict the traffic fluctuations on edges in the considered networks effectively during the evolutionary process, in which $R(t)$ is not larger than $(\frac{1+m\delta+2m}{1+m\delta})^t$. Discussion of traffic and its fluctuation at nodes will subsequently be given in the next section.

3.2 Strength distribution

In a weighted network, a node strength is a natural generalization of its connectivity. The strength s_i of node i is defined as

$$s_i = \sum_{j \in \Omega_i} w_{ij}, \quad (20)$$

where w_{ij} denotes the weight of the link between nodes i and j , Ω_i is the set of all the nearest neighbors of i . The strength distribution $P(s)$ measures the probability that a randomly selected node has exactly strength s .

Let $s_i(t)$ be the strength of node i at step t . If node i is added to the network at step t_i , then $s_i(t_i) = 2$. Moreover, we introduce the quantity $\Delta s_i(t)$, which is defined as the difference between $s_i(t)$ and $s_i(t-1)$. By construction, one can easily obtain

$$\begin{aligned}
\Delta s_i(t) &= s_i(t) - s_i(t-1) = m\delta \sum_{j \in \Omega_i} w_{ij} + m \sum_{j \in \Omega_i} w_{ij} \\
&= m\delta s_i(t-1) + m s_i(t-1).
\end{aligned} \tag{21}$$

Here the first item accounts for the increase of weight of the old links existing in step $t-1$. The second term describes the total weight of the new links with unit weight that are generated in step t and connected to i .

From Eq. (21), one can derive following recursive relation:

$$s_i(t) = (1 + m\delta + m)s_i(t-1). \tag{22}$$

Using $s_i(t_i) = 2$, we obtain

$$s_i(t) = 2(1 + m\delta + m)^{t-t_i}. \tag{23}$$

Since the strength of each node has been obtained explicitly in Eq. (23), we can get the strength distribution via its cumulative distribution [3,43], i.e.

$$P_{\text{cum}}(s) = \sum_{\mu \leq t_i} \frac{n_v(\mu)}{N_t} = \frac{(1 + m\delta + 2m)^{t_i} + \delta + 1}{(1 + m\delta + 2m)^t + \delta + 1}. \tag{24}$$

From Eq. (23), we can derive $t_i = t - \frac{\ln(s/2)}{\ln(1+m\delta+m)}$. Substituting the obtained result of t_i into Eq. (24), we have

$$\begin{aligned}
P_{\text{cum}}(s) &= \frac{(1 + m\delta + 2m)^t \left(\frac{s}{2}\right)^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}} + \delta + 1}{(1 + m\delta + 2m)^t + \delta + 1} \\
&\approx \left(\frac{s}{2}\right)^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}} \quad \text{for large } t.
\end{aligned} \tag{25}$$

Thus, node strength distribution exhibits a power law behavior with the exponent $\gamma_s = 1 + \frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}$. For the special case $m = 1$, Eq. (25) recovers the results previously reported in Ref. [42].

On the other hand, the fluctuation at an arbitrary node i is based on $\frac{s_i}{\sum_{i=1}^N s_i}$. Inserting Eq. (23) and Eq. (9) into Eq. (1), one can obtain the average traffic at an arbitrary node i can be represented as

$$\langle f_i \rangle = \frac{s_i}{2W_t} RM = \frac{(1 + m\delta + m)^{t-t_i}}{3(1 + m\delta + 2m)^t} RM, \tag{26}$$

where t_i denotes the entry time of the node i . Then the standard deviation as a function of $\langle f_i \rangle$ is

$$\begin{aligned}\sigma_i^2 = & \left(\frac{1+m\delta+m}{1+m\delta+2m} \right)^t \frac{(1+m\delta+m)^{-t_i} RM}{3} \\ & + \left(\frac{1+m\delta+m}{1+m\delta+2m} \right)^{2t} \frac{(1+m\delta+m)^{-2t_i} (\Delta^2 + \Delta) M^2}{27}.\end{aligned}\quad (27)$$

It is easy to find that both f_i and σ_i decrease exponentially with t as $\frac{1+m\delta+m}{1+m\delta+2m} < 1$, which is similar with the former result on edges. Although the strength of nodes growing exponentially, the traffic and fluctuation at them still decrease with the growing size of the networks in the case that R is a constant or $R(t) \leq \left(\frac{1+m\delta+2m}{1+m\delta}\right)^t$. In other words, the sufficient condition of keeping the potential traffic fluctuation problems away from the resulting networks is $R(t) \leq \left(\frac{1+m\delta+2m}{1+m\delta}\right)^{\frac{\ln(\frac{N_t(2+\delta)}{3})-\delta-1}{\ln(1+m\delta+2m)}}$ or the average number of walkers is invariable. The novel property is interesting and has not been investigated by previous works [3,31,32,33,34,35]. Therefore, to some extent, this model may provide a paradigm to control the traffic fluctuations and improve transport efficiency of the whole network [36].

3.3 Degree distribution

Similarly to the strength, all simultaneously emerging nodes have the same degree. Let $k_i(t)$ be the degree of node i at step t . If node i is added to the network at step t_i , then by construction $k_i(t_i) = 2$. After that, the degree $k_i(t)$ evolves as

$$k_i(t) = k_i(t-1) + m s_i(t-1), \quad (28)$$

where $m s_i(t-1)$ is the degree increment $\Delta k_i(t)$ of node i at step t . Substituting Eq. (23) into Eq. (28), we have

$$\Delta k_i(t) = 2m (1+m\delta+m)^{t-1-t_i}. \quad (29)$$

Consequently, the degree $k_i(t)$ of node i at time t is

$$k_i(t) = k_i(t_i) + \sum_{\eta=t_i+1}^t \Delta k_i(\eta) = 2 + \frac{2}{\delta+1} \left[(m\delta+1+m)^{t-t_i} - 1 \right]. \quad (30)$$

Analogously to computation of cumulative strength distribution, one can find the cumulative degree distribution

$$\begin{aligned}P_{\text{cum}}(k) = & \frac{(1+m\delta+2m)^t \left[\frac{k}{2}(\delta+1) - \delta \right]^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}} + \delta + 1}{(1+m\delta+2m)^t + \delta + 1} \\ \approx & \left[\frac{k}{2}(\delta+1) - \delta \right]^{-\frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}} \quad \text{for large } t.\end{aligned}\quad (31)$$

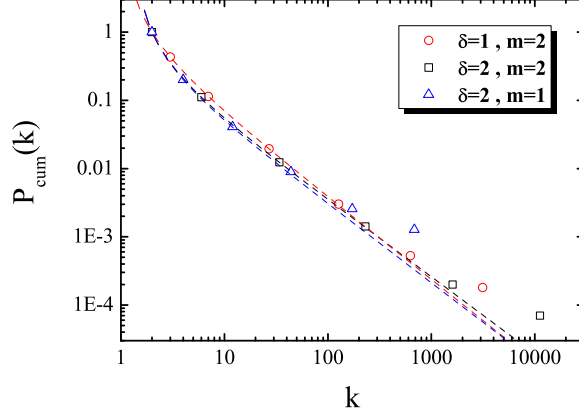


Fig. 2. Cumulative degree distribution $P_{cum}(k)$ versus k for different δ and m corresponding to Eq. (31). The measurements are taken at $t = 5$, illustrating these networks display a power-law degree distribution. The dashed lines are the best fits, with $\gamma_{cum} = 1.20806, -1.12887, -1.15699$ respectively.

As is shown in the Fig. 2, the degree distribution has the scale-free property with the same exponent as γ_s ($\gamma_k = \gamma_s = 1 + \gamma_{cum}$, where $\gamma_{cum} = \frac{\ln(1+m\delta+2m)}{\ln(1+m\delta+m)}$).

3.4 Clustering coefficient

In this model, the analytical expression for clustering coefficient $C(k)$ of the individual node with degree k can be derived exactly. For instance, if a new node is connected to both ends of a link, its degree and clustering coefficient will be 2 and 1, respectively. Naturally, its degree will increase by one when connecting a new node in the next step. On the other hand, there must be an existing neighbor of it attaching to the new node at the same time. On the other hand, there must be an existing neighbor of it, attaching to the new node as well. Because our networks are corresponding to the particular case $q = 2$ of the recursive clique trees [3,4], for a node of degree k , we have

$$C(k) = \frac{1 + (k - 2)}{\frac{k(k-1)}{2}} = 2/k. \quad (32)$$

The scaling $C(k) \sim k^{-1}$ has been found for some network models [7,10,12,17,44], and has also observed in several real-life networks [7].

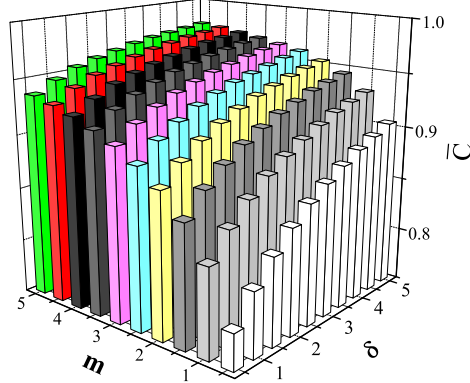


Fig. 3. The solutions of Eq. (33) for δ and m ranging from 1 to 5 respectively. The measurements are taken at $t = 100$, illustrating these networks display a high degree of clustering.

Using Eq. (32), we can obtain the clustering \overline{C}_t of the networks at step t :

$$\overline{C}_t = \frac{1}{N_t} \sum_{r=0}^t \frac{2n_v(r)}{k_r}, \quad (33)$$

where the sum is the total of clustering coefficient for all nodes and $k_r = 2 + \frac{2}{(\delta+1)} [(m\delta + 1 + m)^{t-r} - 1]$ shown by Eq. (30) is the degree of the nodes created at step r .

It can be easily proved that \overline{C}_t increases with q for arbitrary fixed m , and likely \overline{C}_t increases with m when q fixed. In the case of $t = 100$ ($N \rightarrow \infty$), Eq. (33) converges to a correspondingly large value \overline{C} . When $\delta = 2$, for $m = 1, 2, 3, 4$ and 5, \overline{C} equal to 0.886, 0.922, 0.941, 0.952 and 0.96. respectively. When $m = 2$, for $\delta = 1, 2, 3, 4$ and 5, \overline{C} are 0.899, 0.922, 0.937, 0.947 and 0.954, respectively. Evidently, the clustering coefficient of our networks is correspondingly stable and close to 1. Moreover, the average clustering coefficient \overline{C} can be tuned by δ and m (see Fig. 3).

In the classical weighted co-evolutionary models, for example, BBV networks [22,32], the average clustering coefficient rapidly decreases when the networks is growing (see Fig. 4). Our simulations confirm that in the limit of large networks ($N \gg 1$), the BBV networks' clustering coefficient is getting close to zero. However, many real-world networks have a relatively stable and nonzero clustering coefficient, which make the results of BBV model useful, but far from comprehensive. In the Fig. 5, we performed numerical solutions for our model at various values of δ with fixed m in the panel (a) (various values of m with fixed δ in the panel (b)) up to $t = 50$. For the infinite network, one can

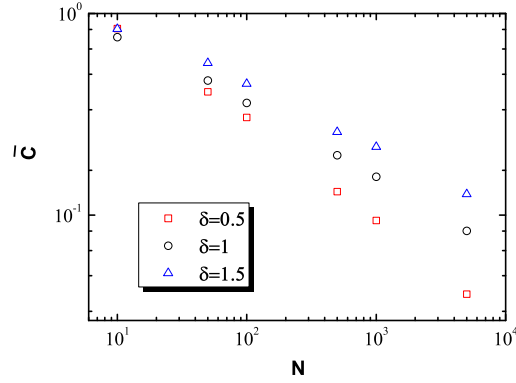


Fig. 4. Average clustering coefficient \overline{C} as a function of size N of nodes for different δ in the BBV networks are shown in the inset, using the weight of a new link $w_0 = 1$, the size of initial seed $m_0 = 3$, the degree of a new node $m = 3$.

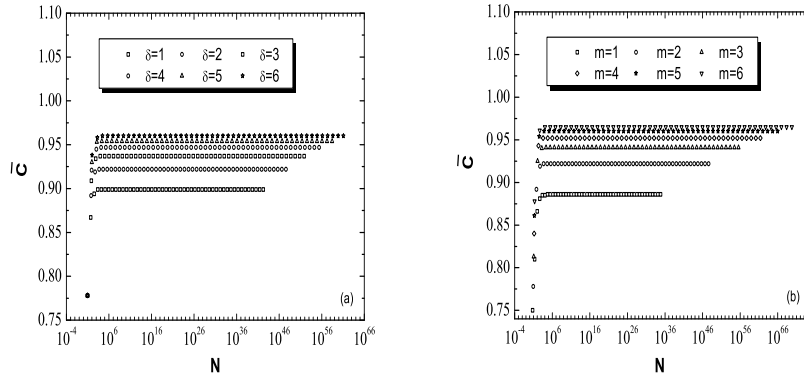


Fig. 5. Semilogarithmic plots of the average clustering coefficient \overline{C} against networks size N for (a) $\delta = 1, 2, \dots, 6$ when $m = 2$ and (b) $m = 1, 2, \dots, 6$ when $\delta = 2$ corresponding to Eq. (33). The measurements are taken over a time interval $t = 1, 2, \dots, 50$, illustrating that \overline{C} is a finite nonzero value independent of N for our model.

easily obtain that this tunable average clustering coefficients of our networks is independent of their size and tends to a nonzero limit, which is a unique property shared by many real networks [6,43]. At the same time, this feature gives excellent agreement with the previous unweighted studies [3].

3.5 Diameter

As is known to all, the diameter of a network is defined as the maximum of the shortest distances between all pairs of nodes, which characterizes the longest communication delay in the network. Small diameter is an important property

of small-world network. Fortunately, for our networks, it can be calculated easily. Below we give the precise analytical computation of diameter of $G(t)$ denoted by $Diam(G(t))$.

It is easy to see that at step $t = 0$ (resp. $t = 1$), the diameter is equal to 1 (resp. 2). At each step $t \geq 2$, one can easily see that the diameter always lies on a pair of nodes created at step t . In order to simplify the analysis, we first note that it is unnecessary to check all the nodes in the networks to fix the diameter. In other words, some nodes (“inner” nodes) added at a given step can be ignored, because they do not increase the diameter of the previous net. Here, so-called “inner” nodes are those that connect to links that already existed before step $t - 1$. Indeed, for these nodes we know that a similar construction has been done in previous steps, so they have nothing to do with the calculation of the diameter.

Let us call “outer” nodes the nodes which are connected to a fresh link. Clearly, at each step, the diameter depends on the distances between outer nodes. At any step $t \geq 2$, we note that an outer node cannot be linked with two nodes created during the same step $r \leq t - 1$. Indeed, we know that from step 2, no outer node is connected to two nodes of the initial triangle $G(0)$. Thus, for any step $t \geq 2$, any outer node is connected with nodes that appeared at pairwise different steps. Now consider two outer nodes created at step $t \geq 2$, say v_t and w_t . Then v_t is connected to two nodes, and one of them must have been created before or during step $t - 2$.

We summarize the above arguments, and gather them into two cases: (a) $t = 2l$ is even. Then, if we make l ‘jumps’ from v_t we reach the initial triangle $G(0)$ in which we can reach any w_t by using a link of $G(0)$ and making l jumps to w_t in a similar way. Thus $Diam(G(2l)) \leq 2l + 1 = t + 1$. (b) $t = 2l + 1$ is odd. In this case we can stop after l jumps at $G(1)$, for which we know that the diameter is 2, and make l jumps in a similar way to reach w_t . Thus $Diam(G(2l + 1)) \leq 2(l + 1) = t + 1$. Obviously, the bound can be reached by pairs of outer nodes created at step t . More precisely, these two nodes v_t and w_t share the property, that both of them are connected to two nodes added at steps $t - 1$, $t - 2$ respectively. Hence, formally, $Diam(G(t)) = t + 1$ for any $t \geq 0$. Considering $N_t \sim (1 + m\delta + 2m)^t$, the diameter is small and scales logarithmically with the number of network nodes.

3.6 Degree correlations

In complex network, degree correlations [39,40,45,46,47], has attracted much attention, because it can give out a unique description of network structures, which could help researchers understand the characteristics of net-

works [39,46,47,48]. An interesting quantity related to degree correlations is the average degree of the nearest neighbors for nodes with degree k , denoted as $k_{nn}(k)$ [46,47,39]. When $k_{nn}(k)$ increases with k , it means that nodes have a tendency to connect to nodes with a similar or larger degree. In this case the network is defined as assortative [39,40]. In contrast, if $k_{nn}(k)$ is decreasing with k , which implies that nodes of large degree are likely to have near neighbors with small degree, then the network is said to be disassortative. If correlations are absent, $k_{nn}(k) = \text{const}$.

In our networks, we can acquire $k_{nn}(k)$ exactly by Eq. (30). Except for three initial nodes generated at step 0, no nodes born in the same step, will be linked to each other. All links from the newcomers to old nodes with larger degree are made at their creation steps. Then, these newcomers become old ones to accept the nodes with smaller degree made at each subsequent steps. These results are shown in the expression

$$k_{nn}(k) = \frac{1}{n_v(t_i)k(t_i, t)} \left(\sum_{t'_i=0}^{t'_i=t_i-1} m \cdot n_v(t'_i) s(t'_i, t_i - 1) k(t'_i, t) \right. \\ \left. + \sum_{t'_i=t_i+1}^{t'_i=t} m \cdot n_v(t_i) s(t_i, t'_i - 1) k(t'_i, t) \right). \quad (34)$$

Here the first sum on the right-hand side accounts for the links made to nodes with larger degree (i.e., $t'_i < t_i$) when the node was generated at t_i . The second sum describes the links made to the current smallest degree nodes at each step $t'_i > t_i$.

Substituting Eqs. (10) and (30) into Eq. (34), one expects that

$$k_{nn}(t_i, t) \approx 2 \left[\frac{(m\delta + 1 + 2m)(m\delta + 1 + m)}{m\delta^2 + 2m\delta + m + \delta} \cdot \frac{(m\delta + 1 + m)^{2t_i}}{(m\delta + 1 + 2m)^{t_i}} + \frac{t - t_i}{m\delta + 1 + m} \right], \quad (35)$$

in the infinite limit of t , where $k_r \approx \frac{2}{\delta+1}(m\delta + 1 + m)^{t-t_i}$. In another word, the initial step $k_{nn}(t_i, t)$ grows linearly with time. Consequently, writing Eq. (35) in terms of k , it is straightforward to obtain

$$k_{nn}(k, t) \approx 2 \frac{(m\delta + 1 + 2m)(m\delta + 1 + m)}{m\delta^2 + 2m\delta + m + \delta} \frac{(m\delta + 1 + m)^{2t}}{(m\delta + 1 + 2m)^t} \\ \cdot \left[\frac{k - 2}{2} (\delta + 1) + 1 \right]^{-\frac{2 \ln(1+m\delta+m) - \ln(1+m\delta+m)}{\ln(1+m\delta+m)}}. \quad (36)$$

Apparently, $k_{nn}(k)$ is approximately a power law function of k with negative exponent, which indicates that the networks are disassortative. Note that $k_{nn}(k)$ of the Internet exhibits a similar power-law dependence on the degree $k_{nn}(k) \sim k^{-w}$, with $w = 0.5$ [46].

4 Conclusion and discussion

To sum up, we have proposed and investigated a deterministic weighted network model, which is constructed in a recursive fashion. The recursive construction guarantees that the traffic fluctuations of nodes and edges decrease exponentially with the time of evolution. The weights of these networks characterizing the various connections exhibit complex statistical features with highly tunable degree, strength, and weight distributions, which display power-law behavior. We have shown the analytical results for degree distributions with tunable exponent and large clustering coefficient, as well as small diameter. Particularly, the features of clustering coefficient in our proposed model, i.e., it is independent of its net size, might lead to a better understanding of realistic networks. To some extent, our model can thus perform well in controlling and designing a variety of weighted scale-free small-world networks to improve their transport efficiency.

Acknowledgment

This research was supported by the National Basic Research Program of China under grant No. 2007CB310806, the National Natural Science Foundation of China under Grant Nos. 60704044, 60873040 and 60873070, Shanghai Leading Academic Discipline Project No. B114, and the Program for New Century Excellent Talents in University of China (NCET-06-0376).

References

- [1] A. L. Barabási, E. Ravasz, T. Vicsek, *Physica A* **299** (2001) 559.
- [2] K. Iguchi, H. Yamada, *Phys. Rev. E* **71** (2005) 036144.
- [3] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes, *Phys. Rev. E* **65** (2002) 066122.
- [4] F. Comellas, G. Fertin, A. Raspaud, *Phys. Rev. E* **69** (2004) 037104.

- [5] S. Jung, S. Kim and B. Kahng, Phys. Rev. E **65** (2002) 056101.
- [6] E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, A. L. Barabási, Science **297** (2002) 1551.
- [7] E. Ravasz, A. L. Barabási, Phys. Rev. E **67** (2003) 026112.
- [8] J. D. Noh, Phys. Rev. E **67** (2003) 045103.
- [9] J. C. Nacher, N. Ueda, M. Kanehisa, T. Akutsu, Phys. Rev. E **71** (2005) 036132.
- [10] J. S. Andrade, H. J. Herrmann, R. F. S. Andrade, L. R. da Silva, Phys. Rev. Lett. **94** (2005) 018702.
- [11] Z. Z. Zhang, L. L. Rong, C. H. Guo, Physica A **363** (2005) 567.
- [12] T. Zhou, B. H. Wang, P. M. Hui, K. P. Chan, Physica A **367** (2006) 613.
- [13] E. M. Bollt, D. ben-Avraham, New J. Phys. **7** (2005) 26.
- [14] M. Hinczewski, A. N. Berker, Phys. Rev. E **73** (2006) 066126.
- [15] Z. Z. Zhang, J. H. Guan, W. L. Xie, Y. Qi, S. G. Zhou, EPL **86** (2009) 10006.
- [16] X. P. Xu, W. Li, F. Liu, Phys. Rev. E **78** (2008) 052103.
- [17] T. Zhou, G. Yan, B. H. Wang, Phys. Rev. E **71** (2005) 046141.
- [18] Z. Z. Zhang, L. L. Rong, S. G. Zhou, Phys. Rev. E **74** (2006) 046105.
- [19] R. F. S. Andrade, H. J. Herrmann, Phys. Rev. E **71** (2005) 056131.
- [20] P. G. Lind, J. A. C. Gallas, H. J. Herrmann, Phys. Rev. E **70** (2004) 056207.
- [21] R. F. S. Andrade, J. G. V. Miranda, Physica A **356** (2005) 1.
- [22] A. Barrat, M. Barthélemy, R. Pastor-Satorras, A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101** (2004) 3747.
- [23] A. Vázquez, R. Pastor-Satorras, A. Vespignani, Phys. Rev. E **65** (2002) 066130.
- [24] B. Kujawski, B. Tadić, G. J. Rodgers, New J. Phys. **9** (2007) 154.
- [25] S. Kwon, S. Yoon, Y. Kim, Phys. Rev. E **77** (2008) 066105.
- [26] C. S. Zhou, J. Kurths, Phys. Rev. Lett. **96** (2006) 164102.
- [27] R. Yang, W. X. Wang, Y. C. Lai, G. R. Chen, Phys. Rev. E **79** (2009) 026112.
- [28] M. Karsai, R. Juhász, F. Iglói, Phys. Rev. E **73** (2006) 036116.
- [29] G. Yan, T. Zhou, J. Wang, Z. Q. Fu, B. H. Wang, Chin. Phys. Lett. **22** (2005) 510.
- [30] T. Zhou, J. Ren, M. Medo, Y. C. Zhang, Phys. Rev. E **76** (2007) 046115.
- [31] A. Barrat, M. Barthélemy, A. Vespignani, Phys. Rev. E **70** (2004) 066149.

- [32] A. Barrat, M. Barthélemy, A. Vespignani, Phys. Rev. Lett. **92** (2004) 228701.
- [33] Y. B. Xie, W. X. Wang, B. H. Wang, Phys. Rev. E **75** (2007) 026111.
- [34] W. X. Wang, B. H. Wang, B. Hu, G. Yan, Q. Ou, Phys. Rev. Lett. **94** (2005) 188702.
- [35] Z. X. Wu, X. J. Xu, Y. H. Wang, Phys. Rev. E **71** (2005) 066124.
- [36] W. E. Leland, M. S. Taquu, W. Willinger, D. V. Wilson, IEEE/ACM Trans. Netw. **2** (1994) 1.
- [37] S. Meloni, J. Góez-Gardeñs, V. Latora, and Y. Moreno, Phys. Rev. Lett. **100** 208701 (2008).
- [38] Y. C. Zhang, S. Zhou, Z. Z. Zhang, J. H. Guan, S. G. Zhou, arXiv: 0902.3160v3.
- [39] M. E. J. Newman, Phys. Rev. Lett. **89** (2002) 208701.
- [40] M. E. J. Newman, Phys. Rev. E **67** (2003) 026126.
- [41] Z. Z. Zhang, L. L. Rong, S. G. Zhou, Physica A **377** (2007) 329.
- [42] S. N. Dorogvtsev, J. F. F. Mendes, AIP Conf. Proc. **776** (2005) 29.
- [43] M. E. J. Newman, SIAM Review **45** (2003) 167.
- [44] L. N. Wang, J. L. Guo, H. X. Yang, T. Zhou, Physica A **388** (2009) 1713.
- [45] S. Maslov, K. Sneppen, Science **296** (2002) 910.
- [46] R. Pastor-Satorras, A. Vázquez, A. Vespignani, Phys. Rev. Lett. **87** (2001) 258701.
- [47] A. Vázquez, R. Pastor-Satorras, A. Vespignani, Phys. Rev. E **65** (2002) 066130.
- [48] C. M. Song, S. Havlin, H. A. Makse, Nature Phys. **2** (2006) 275.